Double autoresonance in two-dimensional dynamical systems

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The phenomenon of double autoresonance in dynamical systems with two degrees of freedom is examined. We analyze the motion of a particle in a two-dimensional centrally symmetric potential, subject to a homogeneous quasiperiodic external field of elliptical polarization. It is shown that if the particle has a sufficiently small initial energy, a double resonance is established when the slowly varying driving frequency approaches the linear resonance frequency. As the driving frequency is changed further, the double resonance is maintained, causing a continuous excitation of the oscillator. When nonlinearity becomes significant, the motion transforms into nearly circular oscillations. The necessary conditions for the persistence of the autoresonance are studied in detail. [S1063-651X(99)00705-9]

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I. INTRODUCTION

The purpose of the presented work is to explore *dynamic autoresonance* (DAR) in integrable systems with two degrees of freedom (2D systems). Autoresonance is known as a persisting phase locking in a nonlinear oscillator subject to a quasiperiodic perturbation. If the oscillator is initially in resonance, adiabatic changes in the driver's frequency lead to self-adjustments of the oscillator energy and frequency such that resonance is maintained. In early studies, DAR was used in the context of relativistic particle acceleration [1–4]. More recent research deals with excitation of atoms [5], dissociation of molecules [6], and further exploration of particle acceleration [7,8]. These applicative studies were supplemented by research intended to broaden the understanding of the DAR phenomenon, such as in the model of two coupled oscillators, where one is linear with a slowly varying frequency [9] and in the model of three weakly interacting nonlinear oscillators whose parameters vary adiabatically [10]. All these problems were reduced to a one degree of freedom (1D) problem via the single resonance approximation.

The study of DAR has provided a basis for the recently developed field of research of *autoresonant wave interaction* (ARWI), which is the wave analog of the DAR phenomenon. The effect is studied theoretically in the context of plasma wave excitation by laser radiation [11], and with regard to excitation of solitons in a nonlinear medium by an external pump wave [12]. The ARWI effect was also found in *mode conversion* of waves, i.e., the interaction of two waves propagating in a weakly nonlinear inhomogeneous medium [13,14]. These studies are based on the assumption of one dimensional inhomogeneity, and also reduce the problem to a 1D dynamical problem. A review of the studies of weakly nonlinear ARWI was given in Ref. [15]. In the most recent works [16], the theory of fully nonlinear multidimensional ARWI is developed by using the averaged variational principle.

All the dynamical studies mentioned above, made use of the single resonance approximation, and obtained a reduction to a 1D dynamical problem. In the present work, we address the problem of DAR in 2D integrable systems, where a double resonance exists, i.e., two resonating perturbative terms are important simultaneously. The model problem is the motion of a particle in a two-dimensional centrally symmetric potential, subject to an elliptically polarized homogeneous quasiperiodic external field. A physical example of this type may be the excitation of classical dipoles by a transverse quasimonochromatic electromagnetic field in a nonlinear medium. Furthermore, the multidimensional DAR problem is essential to the research of the ARWI effect in the cases where a reduction to a 1D dynamical problem cannot be used. Such may be the case of driven multicomponent waves or of driven waves described by partial differential equations of third (or higher) order.

Our presentation will be as follows. In Sec. II we present a brief comparison between the single and double DAR, and in Sec. III we formulate our problem. In Secs. IV–VI we treat the excitation process in three successive stages: the linear excitation stage, the weakly nonlinear excitation stage, and the fully nonlinear excitation stage. Finally, in Sec. VII we present our conclusions.

II. SINGLE VERSUS DOUBLE AUTORESONANCE

A description of DAR in a 1D driven dynamical system [5] can proceed from writing the system's Hamiltonian in the form

$$H(I,\theta,t)=H_0(I)+V[I,\theta,\Phi(t)], \tag{1}$$

where I and θ are the canonical action-angle variables of the unperturbed problem [$H_0(I)$ is the corresponding unperturbed Hamiltonian], while $V[I,\theta,\Phi(t)]$ is a time dependent perturbation ($V\ll H_0$) assumed to be periodic with respect to the phase variable Φ [$V(I,\theta,\Phi+2\pi)=V(I,\theta,\Phi)$]. This perturbation represents an external driving force having *adiabatically varying* frequency $\nu(t)\equiv d\Phi/dt$. We expand the perturbation in Fourier series:

$$V(I,\theta,\Phi)=\sum_{l,n} a_{l,n}(I)e^{i(l\theta+n\Phi)}. \tag{2}$$

In order to study the resonance problem, we leave only a single term, say $a_{l_1,n_1}(I)\exp[i(l_1\theta+n_1\Phi)]$, in the expansion, assuming that the phase in the exponent in this term is nearly stationary, i.e.,

$$\frac{d(l_1\theta+n_1\Phi)}{dt}\approx l_1\omega(I)+n_1\nu(t)\approx 0, \tag{3}$$

where $\omega\equiv dH_0/dI$ is the frequency of the unperturbed system. Note that the resonance condition (3) excludes other resonances, say $l_1'\Omega(I)+n_1'\nu(t)\approx 0$, unless l_1'/n_1' is a multiple of l_1/n_1. Typically, the only important resonant term in the Hamiltonian is that with l_1,n_1 having no common dividers, since the coefficients of the terms with multiple phases in the Fourier expansion fall off rapidly with the increase of the multiplicity. All these arguments lead to the so called *single resonance approximation* Hamiltonian

$$H(I,\theta,t)=H_0(I)+A(I)\cos[l_1\theta+n_1\Phi+p(I)], \tag{4}$$

which, in our case, differs from that studied in the conventional nonlinear resonance problem [17] by a slow time variation of the driving frequency $\nu(t)$. The nontrivial result in this case is that if at some initial time, say t_0, one starts in the vicinity of the resonance, i.e., $l_1\omega[I(t_0)]+n_1\nu(t_0)\approx 0$, then, under certain conditions, the resonance is preserved at later times *despite* the time variation of the driving frequency. In other words, the system is continuously phase locked to the driver despite variation of its parameters. This is the salient signature of the DAR in the 1D system. Since only one term with nearly stationary phase (corresponding to a single continuously satisfied resonance condition) is left in the Hamiltonian, we refer to this case as a *single* DAR.

Now, we generalize from a one degree of freedom problem described above to a driven system with two degrees of freedom (more than two degrees of freedom are treated similarly). Consider a perturbed Hamiltonian of the form

$$H(I_1,I_2,\theta_1,\theta_2,t)=H_0(I_1,I_2)+V[I_1,I_2,\theta_1,\theta_2,\Phi(t)], \tag{5}$$

where (I_1,θ_1) and (I_2,θ_2) are the two pairs of the canonical action-angle variables of the unperturbed (integrable) problem represented by H_0, V is a small perturbation, and Φ is the phase of the driving force. It is also assumed that the driving frequency $\nu(t)\equiv d\Phi/dt$ is a slowly varying function of time. We again expand V in Fourier series:

$$V[I_1,I_2,\theta_1,\theta_2,\Phi(t)]=\sum_{l,m,n} a_{l,m,n}(I_1,I_2)e^{i(l\theta_1+m\theta_2+n\Phi)}, \tag{6}$$

and in studying a resonant problem discard all the terms in the expansion, except those having almost stationary phase factors. One may have a situation, where only one phase (and its multiples) is stationary, say for $l=l_1$, $m=m_1$, and $n=n_1$:

$$l_1\omega_1(I_1,I_2)+m_1\omega_2(I_1,I_2)+n_1\nu(t)\approx 0, \tag{7}$$

$\omega_{1,2}(I_1,I_2)\equiv\partial H_0/\partial I_{1,2}$ being the two main frequencies of the unperturbed problem. In this case, if one starts in resonance at some initial time, the system will stay in resonance for later times, as the actions $I_{1,2}$ self adjust to preserve the phase locking with the driver. However, in contrast to the case of one degree of freedom, in 2D (or higher-dimensional systems), there exists another *generic* possibility of having a double resonance in the system, i.e., for two incommensurate triads l_1,m_1,n_1 and l_2,m_2,n_2 to satisfy two resonance conditions simultaneously,

$$l_1\omega_1(I_1,I_2)+m_1\omega_2(I_1,I_2)+n_1\nu(t)\approx 0,$$
$$l_2\omega_1(I_1,I_2)+m_2\omega_2(I_1,I_2)+n_2\nu(t)\approx 0 \tag{8}$$

at some initial time. Then, if, at later times, the system continues to maintain these two simultaneous resonance conditions despite the variation of the driving frequency $\nu(t)$, we encounter the *double DAR* in the system. The dynamics in this case can be described by the effective *double resonance* Hamiltonian, where one has two nearly stationary terms in the perturbation, i.e.,

$$H(I,\theta,t)=H_0+A\cos(l_1\theta_1+m_1\theta_2+n_1\Phi+q)+B\cos(l_2\theta_1+m_2\theta_2+n_2\Phi+p). \tag{9}$$

Here A, B, q, and p are functions of $I_{1,2}$, and, as previously, we assume that no one of the triads l_1,m_1,n_1 and l_2,m_2,n_2 has common dividers. Note that, in the single DAR in 2D systems, different pairs of actions $I_{1,2}$ can satisfy the resonance condition at a given time, so the actual realization depends on the initial conditions. In contrast, in the double DAR, the actions $I_{1,2}$ at any time are solely determined by the value of the driving frequency $\nu(t)$ at this time. The following sections are devoted to studying an example of the double DAR in a driven 2D centrally symmetrical oscillator system.

III. FORMULATION OF THE PROBLEM

The Hamiltonian of our problem is composed of the Hamiltonian of the unperturbed system H_0 and a small perturbation V:

$$H=H_0+V, \tag{10}$$

$$V\ll H_0, \tag{11}$$

where the unperturbed system is a particle in a centrally symmetric 2D potential well:

$$H_0=\frac{\dot r^2}{2}+\frac{r^2\dot\varphi^2}{2}+U(r). \tag{12}$$

Here r is the radial coordinate, φ is the azimuthal coordinate, $U(r)$ is the centrally symmetric potential, and the particle mass is normalized to 1. It is assumed that for a sufficiently small amplitude, the potential can be expanded as

$$U(r)\cong\frac{\omega_0 r^2}{2}+\frac{ar^4}{4}. \tag{13}$$

All subsequent calculations are performed with the following dimensionless time and radial variables:

$$r \to \frac{r}{\rho}, \quad t \to \frac{t}{\tau}, \tag{14}$$

where ρ and τ are the characteristic length scale and time scale, respectively:

$$\rho = \sqrt{\frac{\omega_0}{|a|}}, \quad \tau = \frac{1}{\omega_0}. \tag{15}$$

Using the scaling (14), expansion (13) assumes the form

$$U(r) \cong \frac{r^2}{2} \pm \frac{r^4}{4}, \tag{16}$$

where the sign depends on the sign of a. Hereafter we shall refer to the plus sign case only.

The perturbation is assumed to be generated by an external uniform force field, having a quasiharmonic time dependence, and an elliptic polarization in the oscillation plane:

$$\vec{F} = \varepsilon_x \hat{x} \cos \Phi(t) + \varepsilon_y \hat{y} \sin \Phi(t), \quad \varepsilon_x, \varepsilon_y \ll 1. \tag{17}$$

Here Φ is the quasiperiodic phase, and we have chosen the x and y directions to be the normal directions of the elliptic polarization. The resulting perturbed part of the Hamiltonian, is given in polar coordinates by

$$V(r,\varphi,t) = \frac{r}{2}[(\varepsilon_x + \varepsilon_y)\cos(\varphi - \Phi(t)) + (\varepsilon_x - \varepsilon_y)\cos(\varphi + \Phi(t))]. \tag{18}$$

We denote the driving frequency by

$$\nu(t) \equiv \dot{\Phi}(t), \tag{19}$$

and assume that ν starts far from 1 (the linear resonance frequency), and passes through 1 at a later time.

Our problem is most conveniently treated in the canonical action-angle variables of the unperturbed problem $I_1, I_2, \theta_1, \theta_2$ (see Appendix A). I_1 is associated with the radial oscillations (i.e., the oscillations in r) and θ_1 describes the phase of these oscillations. I_2 is the angular momentum, and θ_2 describes the phase of the azimuthal motion in the x-y plane. The perturbation can be spectrally decomposed in terms of the action-angle variables yielding (see Appendix B)

$$V = \frac{\varepsilon_x + \varepsilon_y}{2} \sum_{n=-\infty}^{\infty} a_n(I_1, I_2)\cos(n\theta_1 + \theta_2 - \Phi(t)) + \frac{\varepsilon_x - \varepsilon_y}{2} \sum_{n=-\infty}^{\infty} a_n(I_1, I_2)\cos(n\theta_1 + \theta_2 + \Phi(t)). \tag{20}$$

This implies the existence of two families of resonance conditions in our problem:

$$\omega_2 = \pm \nu - n\omega_1, \quad |n| = 0,1,2,\ldots. \tag{21}$$

We show these resonances as lines in the ω_1-ω_2 plane in Fig. 1 for a fixed value of ν.

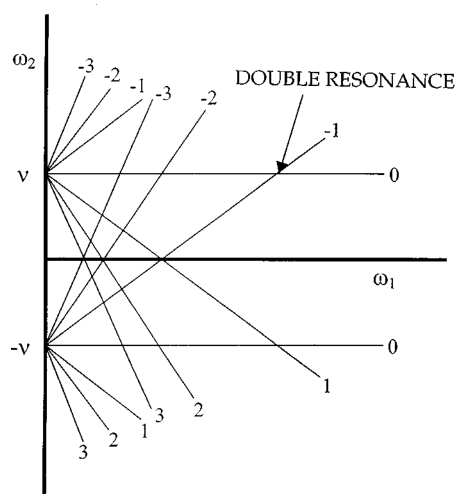

FIG. 1. Resonance lines in the ω_1-ω_2 plane for a fixed value of ν. The numbers correspond to the harmonic index n. The double resonance established in the linear stage is shown by the arrow.

In the following sections we treat the excitation process as a succession of three stages. The initial stage is the linear excitation stage, which is defined by the demand that the nonlinear term in the potential (16) should be smaller than the perturbation (18), and, therefore,

$$r \ll \varepsilon_{x,y}^{1/3}. \tag{22}$$

The second stage is the weakly nonlinear stage, which is defined by the demand that the nonlinear term in the potential (16) should be smaller than the linear term in the potential (16), yet larger than the perturbation (18), implying

$$\varepsilon_{x,y}^{1/3} \ll r \ll 1. \tag{23}$$

The final stage is the fully nonlinear stage, which is defined by the demand that the nonlinear term in the potential (16) should be comparable or larger than the linear term in the potential (16); thus

$$1 \lesssim r. \tag{24}$$

In our problem the oscillator is initially at the linear excitation stage, i.e., inequality (22) is fulfilled.

IV. LINEAR EXCITATION STAGE

We begin the process of particle excitation in the linear stage, where the system can be described as two decoupled linear oscillators. The unperturbed linear Hamiltonian is given by

$$H_0 = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + x^2 + y^2). \tag{25}$$

In Appendix C, we obtain, for the linear case,

$$H_0 = 2I_1 + I_2, \tag{26}$$

$$a_0 = \sqrt{I_1 + I_2}, \quad a_{-1} = -\sqrt{I_1}, \tag{27}$$

$$a_n = 0, \quad n \neq 0, -1,$$

where a_n are the spectral coefficients of expansion (20). The unperturbed Hamiltonian (26) is consistent with the elliptic oscillations, since for each azimuthal cycle there are exactly two radial cycles, i.e.,

$$\omega_1 = \frac{\partial H_0}{\partial I_1} = 2,$$

$$\omega_2 = \frac{\partial H_0}{\partial I_2} = 1. \tag{28}$$

Thus, during the linear oscillation stage only four harmonic terms, with phases $\theta_2 \pm \Phi$ and $\theta_2 - \theta_1 \pm \Phi$, are present in the perturbation. When the driving frequency approaches the linear resonance frequency, two of these terms (with phases $\theta_2 - \Phi$ and $\theta_2 - \theta_1 - \Phi$) become resonant simultaneously. Hence, the system is placed in a double resonance (see Fig. 1). In all subsequent analysis we neglect the nonresonant terms, thus retaining

$$V = \frac{\varepsilon_y + \varepsilon_x}{2}\sqrt{I_1 + I_2}\cos(\theta_2 - \Phi(t))$$

$$+ \frac{\varepsilon_y - \varepsilon_x}{2}\sqrt{I_1}\cos(\theta_1 - \theta_2 - \Phi(t)). \tag{29}$$

We further simplify the expression of the perturbation by introducing the generating function

$$f(\theta_1, \theta_2, I_1', I_2') \equiv I_1'(\theta_1 - \theta_2) + I_2'\theta_2, \tag{30}$$

and transforming to the following action-angle variables:

$$\theta_1' = \theta_1 - \theta_2, \tag{31}$$

$$\theta_2' = \theta_2,$$

$$I_1' = I_1,$$

$$I_2' = I_1 + I_2.$$

The unperturbed linear Hamiltonian (26) and the perturbation (29) in these variables are given by

$$H_0 = I_1' + I_2', \tag{32}$$

$$V = \frac{\varepsilon_1}{2}\sqrt{I_1'}\cos(\theta_1' - \Phi(t)) + \frac{\varepsilon_2}{2}\sqrt{I_2'}\cos(\theta_2' - \Phi(t)), \tag{33}$$

where

$$\varepsilon_1 = \varepsilon_y - \varepsilon_x, \tag{34}$$

$$\varepsilon_2 = \varepsilon_y + \varepsilon_x. \tag{35}$$

The corresponding unperturbed frequencies are

$$\omega_{1,2}' = 1. \tag{36}$$

It was shown in Ref. [13] that in a purely linear system the difference between the phase of the oscillations and the phase of the driving force diminishes as the linear resonance frequency is approached (*phase locking*). Therefore, the linear excitation stage is responsible for establishing a good phase match when nonlinearity appears. It can be shown that in order that phase locking occurs prior to the weakly nonlinear stage, one should demand that the initial driving frequency obeys

$$\varepsilon_{x,y}^{4/3} \ll |\nu(t_0) - 1|. \tag{37}$$

V. WEAKLY NONLINEAR EXCITATION STAGE

A. Approximate Hamiltonian

In this section we develop the approximate Hamiltonian in the weakly nonlinear excitation stage [see inequality (23)], in terms of the action-angle variables. Using Eqs. (12) and (16), we write the unperturbed Hamiltonian in Cartesian coordinates:

$$H_0 = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + x^2 + y^2) + \tfrac{1}{4}(x^4 + y^4 + 2x^2y^2). \tag{38}$$

Definition (23) of the weakly nonlinear stage implies

$$I_{1,2}' \ll 1. \tag{39}$$

Consequently, one obtains the following approximations:

$$a_0 = \sqrt{I_2'} + O(I'^{3/2}), \quad a_{-1} = -\sqrt{I_1'} + O(I'^{3/2}), \tag{40}$$

$$a_k = O(I'^{3/2}), \quad k \neq 0, -1,$$

$$\omega_{1,2}' = 1 + O(I'). \tag{41}$$

The higher order terms in Eqs. (40) and (41) are obtained by a perturbative treatment of the evolution equations, which we do not include here. We use Eqs. (B7), (B8), and (40) to obtain approximate relations between the Cartesian coordinates and the action-angle variables defined by Eq. (31):

$$x = -\sqrt{I_1'}\cos\theta_1' + \sqrt{I_2'}\cos\theta_2' + O(I'^{3/2}), \tag{42}$$

$$y = \sqrt{I_1'}\sin\theta_1' + \sqrt{I_2'}\sin\theta_2' + O(I'^{3/2}). \tag{43}$$

Insertion of Eqs. (42) and (43) into Hamiltonian (38), gives an expression which depends on the action and angle variables. By averaging over the angle variables we obtain the approximate weakly nonlinear Hamiltonian:

$$H_0 = I_1' + I_2' + \frac{I_1'^2}{4} + \frac{I_2'^2}{4} + I_1'I_2' + O(I'^3). \tag{44}$$

B. Autoresonant evolution

We treat our problem in the rotating frame variables

$$\Psi_1 = \theta_1' - \Phi(t) = \theta_1 - \theta_2 - \Phi(t), \tag{45}$$

$$\Psi_2 = \theta_2' - \Phi(t) = \theta_2 - \Phi(t), \tag{46}$$

$$J_1 = I_1' = I_1, \tag{47}$$

$$J_2 = I_2' = I_1 + I_2. \tag{48}$$

Taking the right hand side of Eq. (44) as the unperturbed weakly nonlinear Hamiltonian and the right hand side of Eq. (33) as the perturbation, we have

$$H\cong(J_1+J_2)(1-\nu(t))+\frac{J_1^2}{4}+\frac{J_2^2}{4}+J_1J_2+\frac{\varepsilon_1}{2}\sqrt{J_1}\cos\Psi_1+\frac{\varepsilon_2}{2}\sqrt{J_2}\cos\Psi_2. \quad (49)$$

The DAR solution is composed of a slowly varying quasiequilibrium term and a small oscillating term of $O(\sqrt{\varepsilon})$. We define the quasiequilibrium term as the stationary solution of the evolution equations derived from Hamiltonian (49):

$$\dot{J}_1=\frac{\varepsilon_1}{2}\sqrt{J_1}\sin\Psi_1, \quad (50)$$

$$\dot{J}_2=\frac{\varepsilon_2}{2}\sqrt{J_2}\sin\Psi_2, \quad (51)$$

$$\dot{\Psi}_1=1-\nu(t)+\frac{J_1}{2}+J_2+\frac{\varepsilon_1}{4\sqrt{J_1}}\cos\Psi_1, \quad (52)$$

$$\dot{\Psi}_2=1-\nu(t)+J_1+\frac{J_2}{2}+\frac{\varepsilon_2}{4\sqrt{J_2}}\cos\Psi_2, \quad (53)$$

where we treat $\nu(t)$ as a parameter fixed at a given time moment.

From Eqs. (50) and (51) it follows that the stationary solutions for $\Psi_{1,2}$ are either zero or π. By inspection of the sign of the terms in the right hand side of Eqs. (52) and (53), we conclude that these phases are

$$\Psi_{1,2}^r=\begin{cases}\pi, & \dot{\nu}>0\\ 0, & \dot{\nu}<0.\end{cases} \quad (54)$$

Here the superscript r denotes value in resonance.

At this stage, we introduce the following nomenclature:

$$J_1^r=J\sin^2\alpha, \quad J_2^r=J\cos^2\alpha,$$

$$J>0, \quad 0<\alpha<\frac{\pi}{4}, \quad (55)$$

$$\varepsilon_1=\varepsilon\sin\beta, \quad \varepsilon_2=\varepsilon\cos\beta, \quad (56)$$

$$\varepsilon>0, \quad 0<\beta<\frac{\pi}{4},$$

$$s=\cos\Psi^r, \quad (57)$$

where J and ε describe the amplitude of the nonlinear oscillations and the perturbation, respectively, and $\Psi^r=\Psi_{1,2}^r$. The angles α and β describe the polarization of the nonlinear oscillations and the perturbation, respectively, such that zero angle corresponds to circular polarization, while $\pi/4$ corresponds to linear polarization in the y direction. s is ± 1 according to the two cases in Eq. (54).

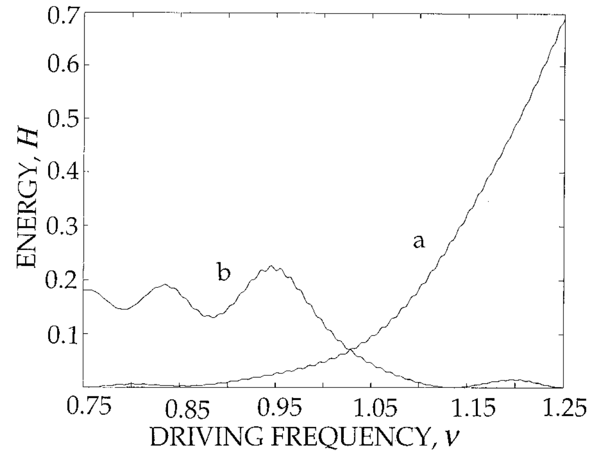

FIG. 2. The numerically found energy vs the driving frequency for the case of increasing driving frequency (curve a) and decreasing driving frequency (curve b).

The resonance line in the (J,α) plane (the locus of resonance points obtained by varying ν), is given by equating the right hand side of Eq. (52) to the right hand side of Eq. (53). After some algebra, one obtains

$$\frac{J^{3/2}}{2\varepsilon}=s\frac{\sin(\alpha-\beta)}{\sin 4\alpha}. \quad (58)$$

We solve Eq. (58) for α, and obtain the following asymptotic solutions:

$$\alpha=\begin{cases}\beta, & J\ll\varepsilon^{2/3},\\ 0, & \varepsilon^{2/3}\ll J, \quad s=-1,\\ \frac{\pi}{4}, & \varepsilon^{2/3}\ll J, \quad s=1.\end{cases} \quad (59)$$

The upper case in Eq. (59) shows that in the linear excitation stage we have oscillations of the same polarization as the perturbation. This can be verified by using the relations

$$y_0+x_0=2\sqrt{J}\cos\alpha, \quad y_0-x_0=2\sqrt{J}\sin\alpha \quad (60)$$

obtained from Eqs. (C6) and (C7). The two lower cases in Eq. (59) indicate that at the weak nonlinearity excitation stage, the oscillations gradually become circular or linear, depending on the sign of $\dot{\nu}$. We examine these possibilities by equating the sum of the right hand side of Eqs. (52) and (53) to zero:

$$\nu(t)-1=\frac{3}{4}J+s\frac{\varepsilon}{\sqrt{2J}}\frac{\sin(\alpha+\beta)}{\sin 2\alpha}. \quad (61)$$

Since for $s=1$ (decreasing driving frequency), the right hand side of Eq. (61) is positive, the resonance will break when $\nu(t)$ decreases below 1. On the other hand, if $s=-1$ (increasing driving frequency), then a solution to Eq. (61) exists for any value of $\nu(t)$ and thus the DAR can be preserved. Depicted in Fig. 2 is the dependence of the energy on the driving frequency, as obtained by solving the equations of motion numerically for both increasing and decreasing driving frequency. In these calculations $\varepsilon_x=0.01$, $\varepsilon_y=0.03$, and

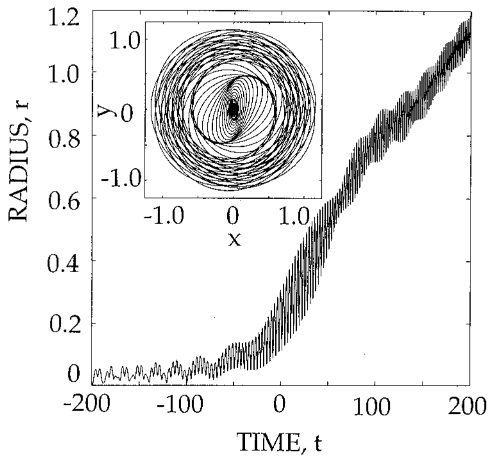

FIG. 3. The radius r as function of time and the orbit in the x-y plane (internal plot) for the case of increasing driving frequency.

$\dot{\nu}=\pm 0.0035$, and initially the oscillator is at rest. One can see that when the driving frequency is increased the energy steadily increases. In contrast, when the driving frequency is decreased the energy ceases to increase in the vicinity of the linear resonance frequency, consistently with the expected resonance breaking. We henceforth treat the case in which the driving frequency is increased. One can see numerical evidence of the transition from elliptical to circular oscillations in Fig. 3, which depicts the radius r as a function of time and the orbit in the x-y plane (internal plot) for the case $\varepsilon_x=0.01$, $\varepsilon_y=0.03$, and $\dot{\nu}=0.0025$.

The resonant actions are obtained by demanding the stationarity of Ψ_1 and Ψ_2, i.e., by equating the right hand sides of Eqs. (52) and (53) to zero. According to Eq. (59), at the weakly nonlinear excitation stage we may use the inequality

$$J_1 \ll J_2, \tag{62}$$

and obtain

$$J_1^r=\left[\frac{\varepsilon_1}{2(1-\nu(t))}\right]^2, \quad J_2^r=2(\nu(t)-1). \tag{63}$$

The stability of the quasiequilibrium solution can be studied by examining the oscillatory part of the DAR solution. Linearizing the evolution equations (50)–(53), in the weakly nonlinear excitation stage, one obtains

$$\delta\dot{J}_1=-\frac{\varepsilon^2\sin^2\beta}{4J}\delta\Psi_1, \tag{64}$$

$$\delta\dot{J}_2=-\frac{\varepsilon\cos\beta}{2}\sqrt{J}\,\delta\Psi_2, \tag{65}$$

$$\delta\dot{\Psi}_1=\frac{J^3}{\varepsilon^2\sin^2\beta}\delta J_1, \tag{66}$$

$$\delta\dot{\Psi}_2=\delta J_1+\frac{\delta J_2}{2}, \tag{67}$$

where $\delta\Psi_1$, $\delta\Psi_2$, δJ_1, and δJ_2 are small deviations of the angles and actions from the quasiequilibrium solution. By diagonalization of the system of equations (64)–(67), one finds that the equilibrium is stable, and the frequencies of the resonant oscillations are

$$\Omega_1=\frac{J}{2}, \quad \Omega_2=\frac{\sqrt{\varepsilon_2}}{2}J^{1/4}. \tag{68}$$

The adiabaticity condition required for the preservation of the autoresonance is given by [5]

$$|\dot{\nu}| \ll \Omega_{1,2}^2. \tag{69}$$

One can also show that

$$|\dot{\nu}| \ll \varepsilon_{x,y}^{4/3} \tag{70}$$

is a sufficient adiabaticity condition all through the weakly nonlinear excitation stage.

The amplitudes of the autoresonant oscillations, which we denote by ΔJ_1, ΔJ_2, $\Delta\Psi_1$, and $\Delta\Psi_2$, are obtained via the following adiabatic relations:

$$\Gamma_k=\frac{\Delta\Psi_k\Delta J_k}{2}=\text{const}, \tag{71}$$

$$\frac{\Delta J_k}{\Delta\Psi_k}=\frac{\varepsilon_k^2}{2J^2}, \quad k=1,2. \tag{72}$$

In order that the DAR will persist, the following inequalities must hold:

$$\Delta J \ll J, \tag{73}$$

$$\Delta\Psi < \pi, \tag{74}$$

where we have omitted the index referring to the degree of freedom. It turns out that the only amplitude whose growth threatens the preservation of the DAR is $\Delta\Psi_1$, which increases according to

$$\Delta\Psi_1=\sqrt{2\Gamma_1}\,\frac{J}{\varepsilon_1}. \tag{75}$$

Thus appropriate initial phase locking is required, in order that the DAR would be preserved throughout the weakly nonlinear excitation stage. Figure 4 depicts the oscillations in Ψ_1 obtained by solving the evolution equations (50)–(53) numerically, for the cases $\varepsilon_x=0.01$, $\varepsilon_y=0.03$, and $\dot{\nu}=0.001$. In this example the phase locking is broken when $\nu(t)=1.56$, due to the increase in $\Delta\Psi_1$.

VI. FULLY NONLINEAR STAGE

A. Approximate Hamiltonian

Next, we turn to the fully nonlinear excitation stage [see inequality (24)]. Assuming that initially in this stage, the particle performs nearly circular oscillations, i.e., that inequality (62) is obeyed, we approximate the unperturbed Hamiltonian by

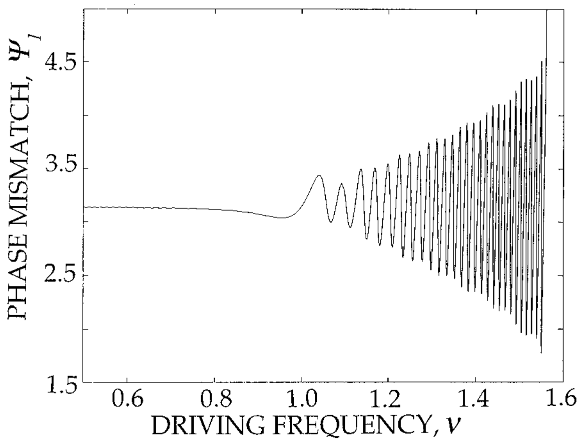

FIG. 4. The mismatch obtained by solving Eqs. (69)–(72) numerically.

$$H_0(J_1,J_2)\cong H_0(J_1=0,J_2)+\tilde{\omega}_1 J_1. \tag{76}$$

The first term on the right hand side of Eq. (76) is the energy of the circular motion, given by [see Eq. (A1)]

$$H_0(J_1=0,J_2)=\frac{J_2^2}{2r_0^2}+U(r_0), \tag{77}$$

where r_0 is the equilibrium radius of the oscillations, which obeys $U'_{\text{eff}}(r_0)=0$, and, therefore, [see Eq. (A3)]

$$J_2^2=r_0^3 U'(r_0). \tag{78}$$

The second term on the right hand side of Eq. (76) is the energy of the small radial oscillations, and $\tilde{\omega}_1$ is the frequency of these oscillations, given by

$$\tilde{\omega}_1(r_0)=\sqrt{U''_{eff}(r_0)}=\sqrt{U''(r_0)+\frac{3U'(r_0)}{r_0}}. \tag{79}$$

Note that according to Hamiltonian (76), the oscillations in Ψ_1 and Ψ_2 are decoupled. Furthermore, the oscillations in Ψ_1 are linear. Consequently, J_1 is approximated by

$$J_1\cong\frac{\tilde{\omega}_1(r_0)\Delta r^2}{2}, \tag{80}$$

where Δr is the small amplitude of the radial oscillations around r_0.

In the fully nonlinear stage we have the same two perturbative resonant terms as in the weakly nonlinear stage [see Eq. (49)]. We may still use relations (27), which are accurate for elliptic oscillations, and obtain approximations for the spectral coefficients:

$$a_0\cong r_0,\quad a_{-1}\cong-\Delta r. \tag{81}$$

B. Autoresonant evolution

According to Sec. IV A the fully nonlinear Hamiltonian may be written as

$$H\cong\frac{J_2^2}{2r_0^2}+U(r_0)-J_2\nu(t)+(\tilde{\omega}_1(r_0)-\nu(t))J_1$$
$$+\frac{\varepsilon_1}{2}\Delta r\cos\Psi_1+\frac{\varepsilon_2}{2}r_0\cos\Psi_2. \tag{82}$$

In order to obtain the evolution equations, we use Eqs. (77)–(80) and obtain the following relations:

$$\frac{\partial r_0}{\partial J_1}=0,$$

$$\frac{\partial r_0}{\partial J_2}=\frac{1}{J_2'},$$

$$\frac{\partial \Delta r}{\partial J_1}=\frac{1}{\tilde{\omega}_1\Delta r},$$

$$\frac{\partial \Delta r}{\partial J_2}=-\Delta r\frac{\tilde{\omega}_1'}{\tilde{\omega}_1 J_2'}, \tag{83}$$

where here $(\cdots)'\equiv\partial(\ldots)/\partial r_0$. Furthermore

$$\frac{\partial H_0}{\partial J_1}=\tilde{\omega}_1,\quad \frac{\partial H_0}{\partial J_2}=\frac{J_2}{r_0^2}. \tag{84}$$

Thus the evolution equations of the fully nonlinear stage are

$$\dot{J}_1=\frac{\varepsilon_1}{2}\Delta r\sin\Psi_1, \tag{85}$$

$$\dot{J}_2=\frac{\varepsilon_2}{2}r_0\sin\Psi_2, \tag{86}$$

$$\dot{\Psi}_1=\tilde{\omega}_1-\nu+\frac{\varepsilon_1}{2\tilde{\omega}_1\Delta r}\cos\Psi_1, \tag{87}$$

$$\dot{\Psi}_2=\frac{J_2}{r_0^2}-\nu, \tag{88}$$

where we neglected $O(\varepsilon)$ terms in the last equation.

We obtain the quasiequilibrium DAR solution as the stationary solution of the evolution equations (85)–(88), where $\nu(t)$ is regarded as a fixed parameter at a given time moment. According to Eqs. (85) and (86), the angles maintain the resonant values of the weakly nonlinear excitation stage [see Eq. (54)]. The radius of oscillations when in resonance, is obtained by the stationarity of Ψ_2 and Eq. (78):

$$\sqrt{\frac{U'(r_0^r)}{r_0^r}}=\nu. \tag{89}$$

We define the *effective exponent* of the central potential by

$$n=\frac{rU''}{U'}+1, \tag{90}$$

so that, for a potential of the form r^k, $n=k$. One can deduce from Eq. (89) that, if $n>2$, i.e., when the potential is increasing faster than the quadratic potential, an increase in the driving frequency will cause an increase in the radius of oscillations. Using Eqs. (79) and (89), we find the following useful relation:

$$\tilde{\omega}_1=\sqrt{n+2}\,\nu. \tag{91}$$

Finally, we obtain an expression for the amplitude of the radial oscillations in the quasiequilibrium solution, through the stationarity of Ψ_1:

$$\Delta r^r=\frac{\varepsilon_1}{2\nu^2\sqrt{n+2}(\sqrt{n+2}-1)}. \tag{92}$$

Here we made use of Eqs. (87) and (91).

Next, we examine the stability of the quasiequilibrium solution by linearizing the evolution equations (85)–(88):

$$\delta\dot{J}_1=-\frac{\varepsilon_1}{2}\Delta r\,\delta\Psi_1, \tag{93}$$

$$\delta\dot{J}_2=-\frac{\varepsilon_2}{2}r_0\,\delta\Psi_2, \tag{94}$$

$$\delta\dot{\Psi}_1=\frac{\varepsilon_1}{2\tilde{\omega}_1^2\Delta r^3}\,\delta J_1, \tag{95}$$

$$\delta\dot{\Psi}_2=\frac{1}{r_0^2}\frac{n-2}{n+2}\,\delta J_2. \tag{96}$$

Here we used relations (78), (80), (83), and (91), and neglected some irrelevant small terms. Equations (93)–(96) describe two decoupled oscillations with frequencies

$$\Omega_1=(\sqrt{n+2}-1)\nu, \quad \Omega_2=\sqrt{\left(\frac{n-2}{n+2}\right)\left(\frac{\varepsilon_2}{2r_0^r}\right)}, \tag{97}$$

where we used Eqs. (91) and (92). In contrast with the weakly nonlinear excitation stage, Ω_2 is expected to decrease during the excitation, and so the adiabaticity condition (69) becomes more restrictive. The amplitudes of the resonant oscillations are obtained in the same manner as in the weakly nonlinear stage [see Eqs. (71) and (72)], whereby one can show that

$$\Delta\Psi_1\propto\nu^{3/2}. \tag{98}$$

Therefore, the resonance in the radial oscillations will eventually be broken. Numerical calculations show that when the resonance in Ψ_1 is broken, the resonance in Ψ_2 persists, and thus the excitation continues.

We conclude with a numerical solution of the equations of motion for the nonlinear potential $U(r)=(r^2/2)-(r^4/4)$, which is the case of the minus sign of Eq. (16). This potential differs from the case of the plus sign by the fact that the frequency of the nonlinear oscillations decreases with the increase of the energy. Thus the DAR excitation is obtained by decreasing the driving frequency. Furthermore, the process differs from the plus sign case by the fact that the potential possesses a separatrix. A numerical calculation of the evolution of the energy vs the driving frequency, until the approach of the separatrix, is displayed in Fig. 5. In this calculation $\varepsilon_x=0.01$, $\varepsilon_y=0.03$, and $\dot{\nu}=-0.002$, and initially the oscillator is in the quasiequilibrium. In Fig. 6 we plot the radius r as function of time and the corresponding orbit in the x-y plane (internal plot) for the same problem. One can see the autoresonant transition from elliptical to circular orbit followed by the escape from the potential, as one approaches the separatrix.

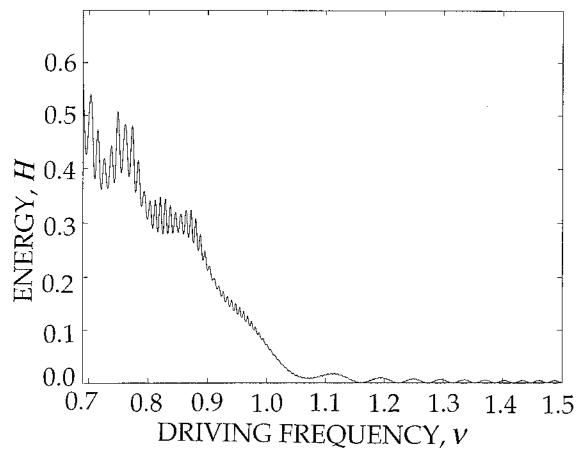

FIG. 5. The autoresonant evolution of the energy vs driving frequency, for the potential $U(r)=(r^2/2)-(r^4/4)$.

VII. CONCLUSIONS

In the presented work, the phenomenon of double autoresonance of a driven 2D oscillator, in a centrally symmetric potential was studied in detail. A uniform, quasiperiodic, elliptically polarized, external field was used as a driver. It was shown that if initially the quartic term in the Hamiltonian is smaller than the perturbation, a double resonance is established when the driving frequency approaches the linear resonance frequency. Using the action-angle representation, the double resonance was decomposed into radial and azi-

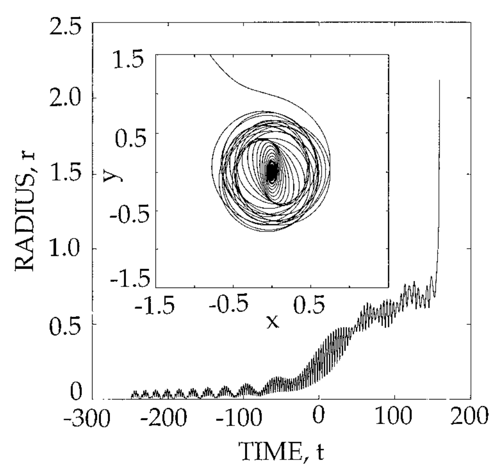

FIG. 6. The radius r as function of time and the orbit in the x-y plane (internal plot) for the potential $U(r)=(r^2/2)-(r^4/4)$.

muthal resonances. Nonlinearity ensures that as the driving frequency is varied, the resonance is maintained, thus causing a continuous excitation of the oscillator. The action-angle variables in this dynamic autoresonance solution comprise a superposition of smooth quasiequilibrium parts and small oscillations.

When the quartic term in the Hamiltonian becomes comparable with the perturbation, the motion transforms into nearly circular oscillations. A perturbative analysis, around the quasiequilibrium DAR solution, has shown the stability of the excitation process. As the excitation continues, the resonant oscillations of the radial angle variable increase, until the radial resonance is broken. Furthermore, the adiabaticity condition on the azimuthal resonance becomes more restrictive as nonlinearity increases.

Additional analysis of the problem when dissipation is present, and, when small deviations from central symmetry are included, will be published elsewhere. There are several other possible generalizations to this study. We have shown that by applying a single frequency, homogeneous, driving field in the problem, a specific combination of two resonances is established. It may be interesting to study in what manner a slowly varying, two frequency, perturbation determines which resonances are established, and what form of oscillations is obtained. Furthermore, one may attempt to extend the work to *n*-dimensional systems. Finally, we mention the possibility of studying the quantum analog of the problem, thus making the analysis more relevant to microscopic oscillators.

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APPENDIX A: ACTION-ANGLE VARIABLES

The treatment of the DAR phenomenon is most conveniently described in terms of action-angle variables of the unperturbed problem. Since our unperturbed system has two degrees of freedom and one cyclic coordinate, it is integrable. Hamiltonian (12) is given in canonical momenta and coordinates by

$$H_0=\frac{p_r^2}{2}+\frac{p_\varphi^2}{2r^2}+U(r), \tag{A1}$$

where p_r and p_φ are the momenta conjugate to r and φ, respectively, given by

$$p_r=\dot r, \quad p_\varphi=r^2\dot\varphi. \tag{A2}$$

Note that p_φ is the angular momentum of the particle, and is an integral of motion. If trapped, the r coordinate of the particle oscillates between two radii $r_{\min}$ and $r_{\max}$, in an effective potential,

$$U_{\text{eff}}(r)=\frac{p_\varphi^2}{2r^2}+U(r), \tag{A3}$$

which depend on the value of the angular momentum.

One can obtain an explicit expression for the orbit in the r-p_r plane:

$$p_r(H_0,p_\varphi,r)=\pm\sqrt{2H_0-\frac{p_\varphi^2}{r^2}-2U(r)}, \tag{A4}$$

where the plus sign corresponds to the $r_{\min}\to r_{\max}$ half of the oscillation, and the minus sign corresponds to the $r_{\max}\to r_{\min}$ half of the oscillation. We use Eq. (A4) to define the canonical action variables

$$I_1(H_0,p_\varphi)\equiv\frac{1}{2\pi}\oint p_r(H_0,p_\varphi,r)dr, \tag{A5}$$

$$I_2(p_\varphi)\equiv\frac{1}{2\pi}\oint p_\varphi d\varphi=p_\varphi, \tag{A6}$$

where the integration is performed along one oscillation of r and φ, respectively, and we have used the fact that φ is a cyclic coordinate. Next, we write the mixed variables generating function:

$$f(r,I_1;\varphi,I_2)\equiv I_2\varphi+\int_{r_{\min}(I_1,I_2)}^{r} p_r(H_0(I_1,I_2),I_2,r')dr', \tag{A7}$$

where here the integration is along the orbit in the r-p_r plane. Next, we use the generating function (A7) to define the corresponding angle variables

$$\theta_1(I_1,I_2,r)\equiv\frac{\partial f}{\partial I_1}=\int_{r_{\min}(I_1,I_2)}^{r}\frac{\partial p_r(I_1,I_2,r')}{\partial I_1}dr', \tag{A8}$$

$$\theta_2(I_1,I_2,\theta_1,\varphi)\equiv\frac{\partial f}{\partial I_2}=\varphi+g(I_1,I_2,\theta_1), \tag{A9}$$

where

$$g(I_1,I_2,\theta_1)\equiv\int_{r_{\min}(I_1,I_2)}^{r(I_1,I_2,\theta_1)}\frac{\partial p_r(I_1,I_2,r')}{\partial I_2}dr', \tag{A10}$$

and $r(I_1,I_2,\theta_1)$ in the right hand side of Eq. (A10) is the periodic function of θ_1, obtained by inversion of Eq. (A8). Note that only positive values of θ_1 have been defined in Eq. (A8). We can continuously extend the definition to negative values, by changing the sign of the integral in the generating function (A7).

The dynamics of the unperturbed system is given by $H=H_0(I_1,I_2)$, yielding the following equations of motion:

$$\dot I_1=0, \tag{A11}$$

$$\dot I_2=0, \tag{A12}$$

$$\dot\theta_1=\omega_1=\frac{\partial H_0(I_1,I_2)}{\partial I_1}, \tag{A13}$$

$$\dot\theta_2=\omega_2=\frac{\partial H_0(I_1,I_2)}{\partial I_2}. \tag{A14}$$

It is understood that θ_1 describes the radial oscillation in the r-p_r plane, and that $\theta_1=0$ corresponds to $r=r_{\min}$ and $\theta_1=\pi$ corresponds to $r=r_{\max}$. θ_2 characterizes the azimuthal oscillation in the x-y plane. One can show that g is a periodic function of θ_1, and that

$$g(I_1,I_2,k\pi)=0, \tag{A15}$$

where k is an integer. Hence, we conclude from Eq. (A9) that φ follows θ_2 with a difference which oscillates with the angle θ_1. When $r=r_{\min}$ or $r=r_{\max}$, θ_2 is equal to φ.

APPENDIX B: SPECTRAL DECOMPOSITION OF THE PERTURBATION

We perform a spectral decomposition of the perturbation, in terms of the action-angle variables of the unperturbed problem by [see Eq. (18)]

$$V=\frac{r(I_1,I_2,\theta_1)}{2}[(\varepsilon_x+\varepsilon_y)\cos(\theta_2-g(I_1,I_2,\theta_1)-\Phi(t)) +(\varepsilon_x-\varepsilon_y)\cos(\theta_2-g(I_1,I_2,\theta_1)+\Phi(t))]. \tag{B1}$$

Taking into account the fact that r and g are periodic functions of θ_1, we expand in a Fourier series:

$$r(I_1,I_2,\theta_1)e^{-ig(I_1,I_2,\theta_1)}=\sum_{n=-\infty}^{\infty} a_n(I_1,I_2)e^{in\theta_1}. \tag{B2}$$

Using definitions (A8) and (A10) one can show that

$$r(I_1,I_2,\theta_1)=r(I_1,I_2,-\theta_1), \tag{B3}$$

$$g(I_1,I_2,\theta_1)=-g(I_1,I_2,-\theta_1). \tag{B4}$$

Hence it follows that

$$\mathrm{Im}(a_n)=0. \tag{B5}$$

We insert Eq. (B2) in Eq. (B1), and obtain the desired spectral decomposition

$$V=\frac{\varepsilon_x+\varepsilon_y}{2}\sum_{n=-\infty}^{\infty} a_n(I_1,I_2)\cos(n\theta_1+\theta_2-\Phi(t)) +\frac{\varepsilon_x-\varepsilon_y}{2}\sum_{n=-\infty}^{\infty} a_n(I_1,I_2)\cos(n\theta_1+\theta_2+\Phi(t)). \tag{B6}$$

By using Eq. (B2), one can also obtain the useful relations

$$x=\sum_{n=-\infty}^{\infty} a_n(I_1,I_2)\cos(n\theta_1+\theta_2), \tag{B7}$$

$$y=\sum_{n=-\infty}^{\infty} a_n(I_1,I_2)\sin(n\theta_1+\theta_2). \tag{B8}$$

APPENDIX C: LINEAR PROBLEM IN ACTION-ANGLE VARIABLES

The linear Hamiltonian is given by [see Eq. (25)]

$$H_0=\frac{1}{2}(\dot{x}^2+\dot{y}^2+x^2+y^2). \tag{C1}$$

The orbits of the unperturbed system in the x-y plane are therefore

$$x(t)=x_0\cos(t-t_0),$$

$$y(t)=y_0\sin(t-t_0). \tag{C2}$$

The trajectory is an ellipse, and we have chosen the x and y axes to be the normal directions of the ellipse. t_0 is taken to be the time in which the particle is on the positive x axis. We further assume, without loss of generality, that

$$0<x_0<y_0 \tag{C3}$$

thus also requiring

$$0<\varepsilon_x<\varepsilon_y. \tag{C4}$$

Since when $r=r_{\min}$, $\theta_1=0$ and $\theta_2=\varphi$, we have

$$\theta_1(t_0)=\theta_2(t_0)=0. \tag{C5}$$

Explicit expressions for the action variables in the linear case are [see definitions (A5) and (A6)]

$$I_1=\frac{(y_0-x_0)^2}{4}, \tag{C6}$$

$$I_2=x_0y_0. \tag{C7}$$

Similarly, we find

$$H_0=2I_1+I_2, \tag{C8}$$

$$\theta_1=2(t-t_0), \tag{C9}$$

$$\theta_2=t-t_0. \tag{C10}$$

Insertion of Eqs. (C9) and (C10) into Eqs. (B7) and (B8), yields expressions for the spectral coefficients:

$$a_0=\frac{y_0+x_0}{2}=\sqrt{I_1+I_2},$$

$$a_{-1}=-\frac{y_0-x_0}{2}=-\sqrt{I_1},$$

$$a_n=0,\quad n\neq 0,-1. \tag{C11}$$

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